

# ON THE MARTINGALE PROPERTY OF CERTAIN LOCAL MARTINGALES

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**ABSTRACT.** The stochastic exponential  $Z_t = \exp\{M_t - M_0 - (1/2)\langle M, M \rangle_t\}$  of a continuous local martingale  $M$  is itself a continuous local martingale. We give a necessary and sufficient condition for the process  $Z$  to be a true martingale in the case where  $M_t = \int_0^t b(Y_u) dW_u$  and  $Y$  is a one-dimensional diffusion driven by a Brownian motion  $W$ . Furthermore, we provide a necessary and sufficient condition for  $Z$  to be a uniformly integrable martingale in the same setting. These conditions are deterministic and expressed only in terms of the function  $b$  and the drift and diffusion coefficients of  $Y$ . As an application we provide a deterministic criterion for the absence of bubbles in a one-dimensional setting.

## 1. INTRODUCTION

The question of whether a local martingale is a strict local martingale<sup>1</sup> or a true martingale is of a particular interest for the stochastic exponential

$$(1) \quad Z_t := \mathcal{E}(M)_t = \exp \left\{ M_t - M_0 - \frac{1}{2} \langle M, M \rangle_t \right\}$$

of a continuous local martingale  $M$  because such a process  $Z$  is often used as a density process for a (locally) absolutely continuous measure change. One can perform the measure change only if  $Z$  is a martingale. The problem of finding convenient sufficient conditions on  $M$  for  $Z$  to be a martingale has attracted significant interest in the literature. The criteria of Novikov and Kazamaki are particularly well-known. Novikov [32] proved that the condition

$$(2) \quad \mathbb{E} \exp \left\{ \frac{1}{2} \langle M, M \rangle_t \right\} < \infty \quad \forall t \in [0, \infty)$$

guarantees that  $Z$  is a martingale. Kazamaki [25] showed that  $Z$  is a martingale provided

$$(3) \quad \exp \left\{ \frac{1}{2} M \right\} \text{ is a submartingale.}$$

Let us note that (3) is equivalent to the condition  $\mathbb{E} \exp\{(1/2)M_t\} < \infty$  for all  $t \in [0, \infty)$  if  $M$  is a true martingale (not just a continuous local martingale as assumed above). Novikov's criterion is of narrower scope but often easier to apply. For improvements on the criteria of Novikov and Kazamaki in the setting of Brownian motion see e.g. Kramkov and Shiryaev [27], Cherny and

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<sup>1</sup>A *strict local martingale* is a local martingale that is not a martingale. This terminology was introduced by Elworthy, Li, and Yor [13].

Shiryaev [5] and the references therein. A similar question in the exponential semimartingale framework has also attracted attention in the literature (see e.g. Kallsen and Shiryaev [23] and the references therein). While (2) and (3) are only sufficient conditions, Engelbert and Senf [17] and, recently, Blei and Engelbert [2] provided necessary and sufficient conditions for  $Z$  to be a martingale. In [17] the case of a general continuous local martingale  $M$  is considered and the condition is given in terms of the time-change that turns  $M$  into a (possibly stopped) Brownian motion. In [2] the case of a strong Markov continuous local martingale  $M$  is studied and the deterministic criterion is expressed in terms of the speed measure of  $M$ . In the recent papers of Kotani [26] and Hulley and Platen [19] a related question is studied. These authors obtain necessary and sufficient conditions for a one-dimensional regular strong Markov continuous local martingale to be a martingale. For further literature review see e.g. the bibliographical notes in the monographs Karatzas and Shreve [24, Ch. 3], Liptser and Shiryaev [29, Ch. 6], Protter [34, Ch. III], and Revuz and Yor [35, Ch. VIII].

In the present paper we consider local martingales  $M$  of the form  $M_t = \int_0^t b(Y_u) dW_u$ , where  $Y$  is a one-dimensional diffusion driven by a Brownian motion  $W$ . Our main results are necessary and sufficient conditions for  $Z$  to be a true martingale (Theorem 2.1) and for  $Z$  to be a uniformly integrable martingale (Theorem 2.3). The conditions are deterministic and expressed only in terms of the function  $b$  and the drift and diffusion coefficients of  $Y$ .

Compared with the aforementioned result of [17], our criterion is of narrower scope (Engelbert and Senf consider an arbitrary continuous local martingale  $M$ ), but our results are easier to apply when  $M_t = \int_0^t b(Y_u) dW_u$ , as the condition of Engelbert and Senf is given in terms of the Dambis–Dubins–Schwarz Brownian motion of  $M$  and the related time-change. The setting of the present paper differs from that of [2], [26], and [19] in that in our case the process  $Y$  possesses the strong Markov property but both  $M = \int_0^\cdot b(Y_u) dW_u$  and  $Z = \mathcal{E}(M)$  can well be non-Markov. As a simple example, consider  $Y \equiv W$  and  $b(x) = I(x > 0)$ . Then both  $M$  and  $Z$  have intervals of constancy and therefore the knowledge of the trajectory's past helps to predict the future in the following way. Let us fix a time  $t$  and a position  $M_t$  (resp.  $Z_t$ ). If  $M$  (resp.  $Z$ ) is constant immediately before  $t$ ,  $W$  has a negative excursion at time  $t$ , hence  $M$  (resp.  $Z$ ) will be constant also immediately after  $t$ . Similarly, if  $M$  or  $Z$  is oscillating immediately before  $t$ , then it will be oscillating also immediately after  $t$ . This implies that  $M$  and  $Z$  are non-Markov.

We discuss the applications of Theorems 2.1 and 2.3 in specific situations by studying several examples. There is evidence in the literature that in some settings the loss of the martingale property of the stochastic exponential  $Z$  is related to the explosion of some auxiliary diffusion (see Karatzas and Shreve [24, Ch. 5, Ex. 5.38], Revuz and Yor [35, Ch. IX, Ex. (2.10)], Sin [36], Carr, Cherny, and Urusov [3]). Such a statement turns out to be true in the case where the diffusion  $Y$  is itself non-explosive (see Crollary 2.2) but fails in general (see Example 3.1). The loss of the martingale property of  $Z$  is in our setting related only to the explosion of the auxiliary diffusion at a *bad endpoint* (this notion is defined in the next section). As another application of Theorems 2.1 and 2.3, we obtain a deterministic necessary and sufficient condition for the absence of bubbles in diffusion-based models.

The paper is structured as follows. In Section 2 we describe the setting and formulate the main results, which are proved in Section 6. In Section 3 we illustrate the main results by a complete study of two examples. Applications to financial bubbles are given in Section 4. In Section 5 we recall the definition of *separating time* (see [6]) and formulate the results about separating times that are used in the proofs in Section 6. Finally, in the Appendix we state some well-known facts about the behaviour of one-dimensional diffusions that are used extensively in Sections 5.2 and 6.

## 2. MAIN RESULTS

We consider the state space  $J = (l, r)$ ,  $-\infty \leq l < r \leq \infty$  and a  $J$ -valued diffusion  $Y = (Y_t)_{t \in [0, \infty)}$  on some filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$  governed by the SDE

$$(4) \quad dY_t = \mu(Y_t) dt + \sigma(Y_t) dW_t, \quad Y_0 = x_0 \in J,$$

where  $W$  is an  $(\mathcal{F}_t)$ -Brownian motion and  $\mu, \sigma: J \rightarrow \mathbb{R}$  are Borel functions satisfying the Engelbert–Schmidt conditions

$$(5) \quad \sigma(x) \neq 0 \quad \forall x \in J,$$

$$(6) \quad \frac{1}{\sigma^2}, \quad \frac{\mu}{\sigma^2} \in L^1_{\text{loc}}(J).$$

$L^1_{\text{loc}}(J)$  denotes the class of locally integrable functions, i.e. the functions  $J \rightarrow \mathbb{R}$  that are integrable on compact subsets of  $J$ . Under conditions (5) and (6) SDE (4) has a unique in law (possibly explosive) weak solution (see [15], [16], or [24, Ch. 5, Th. 5.15]). We denote the explosion time of  $Y$  by  $\zeta$ . In the case  $\mathbb{P}(\zeta < \infty) > 0$  we need to specify the behaviour of  $Y$  after explosion. In what follows we assume that the solution  $Y$  on the set  $\{\zeta < \infty\}$  stays after  $\zeta$  at the boundary point of  $J$  at which it explodes, i.e.  $l$  and  $r$  become absorbing boundaries. We will use the following terminology:

$Y$  explodes at  $r$  means  $\mathbb{P}(\zeta < \infty, \lim_{t \uparrow \zeta} Y_t = r) > 0$ ;

$Y$  explodes at  $l$  is understood in an analogous way.

Finally, let us note that the Engelbert–Schmidt conditions are reasonable weak assumptions. For instance, they are satisfied if  $\mu$  is locally bounded on  $J$  and  $\sigma$  is locally bounded away from zero on  $J$ .

In this section we consider the stochastic exponential

$$(7) \quad Z_t = \exp \left\{ \int_0^{t \wedge \zeta} b(Y_u) dW_u - \frac{1}{2} \int_0^{t \wedge \zeta} b^2(Y_u) du \right\}, \quad t \in [0, \infty),$$

where we set  $Z_t := 0$  for  $t \geq \zeta$  on  $\{\zeta < \infty, \int_0^\zeta b^2(Y_u) du = \infty\}$ . In what follows we assume that  $b$  is a Borel function  $J \rightarrow \mathbb{R}$  satisfying

$$(8) \quad \frac{b^2}{\sigma^2} \in L^1_{\text{loc}}(J).$$

Below we show that condition (8) is equivalent to

$$(9) \quad \int_0^t b^2(Y_u) du < \infty \quad \mathbb{P}\text{-a.s. on } \{t < \zeta\}, \quad t \in [0, \infty).$$

Condition (9) ensures that the stochastic integral  $\int_0^t b(Y_u) dW_u$  is well-defined on  $\{t < \zeta\}$  and, as mentioned above, is equivalent to imposing (8) on the function  $b$ . Thus the process  $Z = (Z_t)_{t \in [0, \infty)}$  defined in (7) is a nonnegative continuous local martingale (continuity at time  $\zeta$  on the set  $\{\zeta < \infty, \int_0^\zeta b^2(Y_u) du = \infty\}$  follows from the Dambis–Dubins–Schwarz theorem; see [35, Ch. V, Th. 1.6]).

*Discussion.* Since  $Z$  is a nonnegative local martingale, it is a supermartingale (by Fatou's lemma). Hence, for a fixed  $T \in (0, \infty)$ ,  $Z$  is a martingale on the time interval  $[0, T]$  if and only if

$$(10) \quad EZ_T = 1.$$

As a nonnegative supermartingale,  $Z$  has a ( $\mathbb{P}$ -a.s.) limit  $Z_\infty := \lim_{t \uparrow \infty} Z_t$  and is closed by  $Z_\infty$ , i.e. the process  $(Z_t)_{t \in [0, \infty]}$  (with time  $\infty$  included) is a supermartingale. Hence,  $Z$  is a uniformly integrable martingale if and only if (10) holds for  $T = \infty$ . In Theorem 2.1 below (see also the remark following the theorem) we present a deterministic criterion in terms of  $\mu$ ,  $\sigma$ , and  $b$  for (10) with  $T \in (0, \infty)$ . In Theorem 2.3 we give a deterministic necessary and sufficient condition for (10) with  $T = \infty$ .

Before we formulate the results let us show that (8) is equivalent to (9). By the occupation times formula we have

$$(11) \quad \begin{aligned} \int_0^{t \wedge \zeta} b^2(Y_u) du &= \int_0^{t \wedge \zeta} \frac{b^2}{\sigma^2}(Y_u) d\langle Y, Y \rangle_u \\ &= \int_J \frac{b^2}{\sigma^2}(y) L_{t \wedge \zeta}^y(Y) dy, \quad t \in [0, \infty), \end{aligned}$$

where  $L_t^y(Y)$  denotes a continuous in  $t$  and càdlàg in  $y$  version of the local time of  $Y$  at time  $t$  and level  $y$ . Now (9) follows from (8) and the fact that on  $\{t < \zeta\}$  the function  $y \mapsto L_{t \wedge \zeta}^y(Y)$  is càdlàg (hence bounded) with a compact support in  $J$ . Conversely, suppose that (8) is not satisfied. Then there exists a point  $\alpha \in J$  such that we either have

$$(12) \quad \int_\alpha^{\alpha+\varepsilon} \frac{b^2}{\sigma^2}(y) dy = \infty \text{ for any } \varepsilon > 0$$

or

$$(13) \quad \int_{\alpha-\varepsilon}^\alpha \frac{b^2}{\sigma^2}(y) dy = \infty \text{ for any } \varepsilon > 0.$$

Below we assume (12) and  $\alpha < x_0$  (recall that  $x_0$  is the starting point for  $Y$ ). Let us take some  $\beta < \alpha$ ,  $\beta \in J$  and consider the stopping time

$$(14) \quad T_\beta(Y) = \inf\{t \in [0, \infty) : Y_t = \beta\}$$

with the usual convention  $\inf \emptyset := \infty$ . It is well-known that  $\mathbb{P}(T_\beta(Y) < \zeta) > 0$  and the function  $y \mapsto L_{t \wedge \zeta}^y(Y)$  is bounded away from zero on  $(\alpha, x_0)$   $\mathbb{P}$ -a.s. on  $\{T_\beta(Y) < \zeta, t \geq T_\beta(Y)\}$ . Now it follows from (11) and (12) that  $\int_0^t b^2(Y_u) du = \infty$   $\mathbb{P}$ -a.s. on  $\{T_\beta(Y) \leq t < \zeta\}$ , which contradicts (9). Other possibilities ((12) and  $\alpha \geq x_0$ , (13) and  $\alpha < x_0$ , (13) and  $\alpha \geq x_0$ ) can be dealt with in a similar way.

Let us now consider an auxiliary  $J$ -valued diffusion  $\tilde{Y}$  governed by the SDE

$$(15) \quad d\tilde{Y}_t = (\mu + b\sigma)(\tilde{Y}_t) dt + \sigma(\tilde{Y}_t) d\tilde{W}_t, \quad \tilde{Y}_0 = x_0,$$

on some probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \in [0, \infty)}, \tilde{\mathbb{P}})$ . SDE (15) has a unique in law (possibly explosive) weak solution because the Engelbert–Schmidt conditions (5) and (6) are satisfied for the coefficients  $\mu + b\sigma$  and  $\sigma$  (note that  $b/\sigma \in L^1_{\text{loc}}(J)$  due to (8)). Similarly to the case of the solution of SDE (4) we denote the explosion time of  $\tilde{Y}$  by  $\tilde{\zeta}$  and apply the following convention: on the set  $\{\tilde{\zeta} < \infty\}$  the solution  $\tilde{Y}$  stays after  $\tilde{\zeta}$  at the boundary point at which it explodes.

Let  $\overline{J} := [l, r]$ . Let us fix an arbitrary  $c \in J$  and set

$$(16) \quad \rho(x) := \exp \left\{ - \int_c^x \frac{2\mu}{\sigma^2}(y) dy \right\}, \quad x \in J,$$

$$(17) \quad \tilde{\rho}(x) := \rho(x) \exp \left\{ - \int_c^x \frac{2b}{\sigma}(y) dy \right\}, \quad x \in J,$$

$$(18) \quad s(x) := \int_c^x \rho(y) dy, \quad x \in \overline{J},$$

$$(19) \quad \tilde{s}(x) := \int_c^x \tilde{\rho}(y) dy, \quad x \in \overline{J}.$$

Note that  $s$  (resp.  $\tilde{s}$ ) is the scale function of diffusion (4) (resp. (15)). Further, we set

$$\tilde{v}(x) := \int_c^x \frac{\tilde{s}(x) - \tilde{s}(y)}{\tilde{\rho}(y)\sigma^2(y)} dy \quad \text{and} \quad v(x) := \int_c^x \frac{s(x) - s(y)}{\rho(y)\sigma^2(y)} dy, \quad x \in J.$$

Note that the functions  $\tilde{v}$  and  $v$  are decreasing on  $(l, c)$  and increasing on  $(c, r)$ . Therefore the quantities

$$\tilde{v}(r) := \lim_{x \uparrow r} \tilde{v}(x), \quad v(r) := \lim_{x \uparrow r} v(x), \quad \tilde{v}(l) := \lim_{x \downarrow l} \tilde{v}(x), \quad v(l) := \lim_{x \downarrow l} v(x)$$

are well-defined. By  $L^1_{\text{loc}}(r-)$  we denote the class of Borel functions  $f: J \rightarrow \mathbb{R}$  such that  $\int_x^r |f(y)| dy < \infty$  for some  $x \in J$ . Similarly we introduce the notation  $L^1_{\text{loc}}(l+)$ .

Let us recall that the process  $\tilde{Y}$  (resp.  $Y$ ) explodes at the boundary point  $r$  if and only if

$$(20) \quad \tilde{v}(r) < \infty \quad (\text{resp. } v(r) < \infty).$$

This is Feller's test for explosions (see [24, Ch. 5, Th. 5.29]). Sometimes it can be easier to check the following condition

$$(21) \quad \tilde{s}(r) < \infty \text{ and } \frac{\tilde{s}(r) - \tilde{s}}{\tilde{\rho}\sigma^2} \in L^1_{\text{loc}}(r-) \quad (\text{resp. } s(r) < \infty \text{ and } \frac{s(r) - s}{\rho\sigma^2} \in L^1_{\text{loc}}(r-)),$$

equivalent to (20) (see [4, Sec. 4.1]). Similarly,  $\tilde{Y}$  (resp.  $Y$ ) explodes at the boundary point  $l$  if and only if

$$(22) \quad \tilde{v}(l) < \infty \quad (\text{resp. } v(l) < \infty).$$

Sometimes it can be easier to check the equivalent condition

$$(23) \quad \tilde{s}(l) > -\infty \text{ and } \frac{\tilde{s} - \tilde{s}(l)}{\tilde{\rho}\sigma^2} \in L^1_{\text{loc}}(l+) \quad (\text{resp. } s(l) > -\infty \text{ and } \frac{s - s(l)}{\rho\sigma^2} \in L^1_{\text{loc}}(l+)).$$

We say that the endpoint  $r$  of  $J$  is *good* if

$$(24) \quad s(r) < \infty \text{ and } \frac{(s(r) - s)b^2}{\rho\sigma^2} \in L_{\text{loc}}^1(r-).$$

Sometimes it can be easier to check the equivalent condition

$$(25) \quad \tilde{s}(r) < \infty \text{ and } \frac{(\tilde{s}(r) - \tilde{s})b^2}{\tilde{\rho}\sigma^2} \in L_{\text{loc}}^1(r-).$$

We say that the endpoint  $l$  of  $J$  is *good* if

$$(26) \quad s(l) > -\infty \text{ and } \frac{(s - s(l))b^2}{\rho\sigma^2} \in L_{\text{loc}}^1(l+).$$

Sometimes it can be easier to check the equivalent condition

$$(27) \quad \tilde{s}(l) > -\infty \text{ and } \frac{(\tilde{s} - \tilde{s}(l))b^2}{\tilde{\rho}\sigma^2} \in L_{\text{loc}}^1(l+).$$

The equivalence between (24) and (25) as well as between (26) and (27) will follow from remark (iii) after Theorem 5.5. If  $l$  or  $r$  is not good, we call it *bad*.

*Remark.* Even though conditions (24) and (25) are equivalent, the inequalities  $s(r) < \infty$  and  $\tilde{s}(r) < \infty$  are not equivalent. The same holds for conditions (26) and (27).

*Important remark.* The notions of good and bad endpoints of  $J$  are central to the theorems below. To apply these theorems we need to check whether the endpoints are good in concrete situations. In Section 5 (see remark (iv) following Theorem 5.5) we show that the endpoint  $r$  is bad whenever one of the processes  $Y$  and  $\tilde{Y}$  explodes at  $r$  and the other does not. This is helpful because one can sometimes immediately see that, for example,  $Y$  does not explode at  $r$  while  $\tilde{Y}$  does. In such a case one can conclude that  $r$  is bad without having to check either (24) or (25). The same holds for the endpoint  $l$ .

**Theorem 2.1.** *Let the functions  $\mu$ ,  $\sigma$ , and  $b$  satisfy conditions (5), (6) and (8), and  $Y$  be a (possibly explosive) solution of SDE (4). Then the process  $Z$  given by (7) is a martingale if and only if at least one of the conditions (a)–(b) below is satisfied AND at least one of the conditions (c)–(d) below is satisfied:*

- (a)  $\tilde{Y}$  does not explode at  $r$ , i.e. (20) (equivalently, (21)) is not satisfied;
- (b)  $r$  is good, i.e. (24) (equivalently, (25)) is satisfied;
- (c)  $\tilde{Y}$  does not explode at  $l$ , i.e. (22) (equivalently, (23)) is not satisfied;
- (d)  $l$  is good, i.e. (26) (equivalently, (27)) is satisfied.

*Remark.* The same condition is necessary and sufficient for  $Z$  to be a martingale on the time interval  $[0, T]$  for any fixed  $T \in (0, \infty)$  (see the proof of Theorem 2.1 in Section 6).

The case of a non-explosive  $Y$  is of particular interest. In this case Theorem 2.1 takes a simpler form.

**Corollary 2.2.** *Let the assumptions of Theorem 2.1 be satisfied and  $Y$  be non-explosive. Then  $Z$  is a martingale if and only if  $\tilde{Y}$  does not explode.*

This corollary follows immediately from Theorem 2.1 and the important remark preceding it.

**Theorem 2.3.** *Under the assumptions of Theorem 2.1 the process  $Z$  is a uniformly integrable martingale if and only if at least one of the conditions (A)–(D) below is satisfied:*

- (A)  $b = 0$  a.e. on  $J$  with respect to the Lebesgue measure;
- (B)  $r$  is good and  $\tilde{s}(l) = -\infty$ ;
- (C)  $l$  is good and  $\tilde{s}(r) = \infty$ ;
- (D)  $l$  and  $r$  are good.

*Remark.* Condition (A) in Theorem 2.3 cannot be omitted. Indeed, if  $J = \mathbb{R}$ ,  $b \equiv 0$  and  $Y := W$  is a Brownian motion, then  $Z \equiv 1$  is a uniformly integrable martingale but none of the conditions (B), (C), (D) hold because  $s(-\infty) = -\infty$  and  $s(\infty) = \infty$  (and hence neither endpoint is good).

### 3. EXAMPLES

**Example 3.1.** Let us fix  $\alpha > -1$  and consider the state space  $J = (-\infty, \infty)$  and a diffusion  $Y$  governed by the SDE

$$(28) \quad dY_t = |Y_t|^\alpha dt + dW_t, \quad Y_0 = x_0.$$

For each  $\alpha > -1$ , the coefficients of (28) satisfy the Engelbert–Schmidt conditions (see (5)–(6)). Hence, SDE (28) has a unique in law (possibly explosive) weak solution. We are interested in the stochastic exponential

$$Z_t = \exp \left\{ \int_0^{t \wedge \zeta} Y_u dW_u - \frac{1}{2} \int_0^{t \wedge \zeta} Y_u^2 du \right\}, \quad t \in [0, \infty),$$

where we set  $Z_t := 0$  for  $t \geq \zeta$  on  $\{\zeta < \infty, \int_0^\zeta Y_u^2 du = \infty\}$  (like in Section 2,  $\zeta$  denotes the explosion time of  $Y$ ).

Let us now apply the results of Section 2 to study the martingale property of  $Z$  (i.e. we have  $\mu(x) = |x|^\alpha$ ,  $\sigma(x) \equiv 1$ , and  $b(x) = x$ ). We get the following classification:

- (1) If  $-1 < \alpha \leq 1$ , then  $Z$  is a martingale (but not uniformly integrable);
- (2) If  $1 < \alpha \leq 3$ , then  $Z$  is a strict local martingale;
- (3) If  $\alpha > 3$ , then  $Z$  is a uniformly integrable martingale.

Let us now outline some key steps in getting these results (computations are omitted). The auxiliary diffusion  $\tilde{Y}$  is in our case given by the SDE

$$d\tilde{Y}_t = (|\tilde{Y}_t|^\alpha + \tilde{Y}_t) dt + d\tilde{W}_t, \quad \tilde{Y}_0 = x_0.$$

We have:

- (1)  $\tilde{Y}$  does not explode at  $-\infty$  (for each  $\alpha > -1$ );
- (2) If  $-1 < \alpha \leq 1$ , then  $\tilde{Y}$  does not explode at  $\infty$ ;
- (3) If  $\alpha > 1$ , then  $\tilde{Y}$  explodes at  $\infty$ ;
- (4)  $s(-\infty) = -\infty$ , hence, the endpoint  $-\infty$  is bad (for each  $\alpha > -1$ );
- (5) The endpoint  $\infty$  is good if and only if  $\alpha > 3$ ;

(6) If  $\alpha > 3$ , then  $\tilde{s}(-\infty) = -\infty$ .<sup>2</sup>

This is sufficient for the application of the results in Section 2, which yield the classification above.

*Remarks.* (i) It is not surprising to see a qualitative change in the properties of  $Z$  as  $\alpha$  passes level 1 because a qualitative change in the properties of  $Y$  occurs ( $Y$  is non-explosive for  $\alpha \leq 1$  and explosive for  $\alpha > 1$ ). However, it is much more surprising to see a qualitative change in the properties of  $Z$  as  $\alpha$  passes level 3, while  $Y$  undergoes no qualitative change.

One can understand what happens for  $\alpha > 3$  as follows. Let us recall that  $Z$  is a martingale if and only if it does not lose mass at finite times, and is a uniformly integrable martingale if and only if it does not lose mass at time infinity (see (10)). Informally, when  $\alpha > 3$ , the process  $Y$  explodes “so quickly” that  $Z$  does not lose mass.

(ii) There is some evidence in the literature (see Karatzas and Shreve [24, Ch. 5, Ex. 5.38], Revuz and Yor [35, Ch. IX, Ex. (2.10)], Sin [36], Carr, Cherny, and Urusov [3]) that the loss of the martingale property of  $Z$  in some settings is related to the explosion of the auxiliary diffusion  $\tilde{Y}$ . The example studied above shows that this is no longer true in our setting. In the case  $\alpha > 3$  the auxiliary diffusion  $\tilde{Y}$  is explosive, while  $Z$  is a uniformly integrable martingale. As we can see, the loss of the martingale property of  $Z$  is related only to the explosion of  $\tilde{Y}$  at a bad endpoint.

**Example 3.2** (Measure transformation in a generalised CEV process). We now turn to a generalised constant elasticity of variance (CEV) process that satisfies the SDE

$$(29) \quad dY_t = \mu_0 Y_t^\alpha dt + \sigma_0 Y_t^\beta dW_t, \quad Y_0 = x_0 \in J := (0, \infty), \quad \alpha, \beta \in \mathbb{R}, \mu_0 \in \mathbb{R} \setminus \{0\}, \sigma_0 > 0$$

under a probability measure  $\mathsf{P}$ . Note that for the chosen domain  $J$  and the specified parameter ranges the coefficients in (29) are locally Lipschitz, and therefore equation (29) has a pathwise unique strong solution up to explosion time  $\zeta$  (see [35, Ch. IX, Ex. (2.10)]). We are interested in the stochastic exponential

$$(30) \quad Z_t = \exp \left\{ -\frac{\mu_0}{\sigma_0} \int_0^{t \wedge \zeta} Y_u^{\alpha-\beta} dW_u - \frac{1}{2} \frac{\mu_0^2}{\sigma_0^2} \int_0^{t \wedge \zeta} Y_u^{2\alpha-2\beta} du \right\}, \quad t \in [0, \infty),$$

where we set  $Z_t := 0$  for  $t \geq \zeta$  on  $\{\zeta < \infty, \int_0^\zeta Y_u^{2\alpha-2\beta} du = \infty\}$ . If  $Z$  is a martingale (resp. uniformly integrable martingale), then we can perform a locally (resp. globally) absolutely continuous measure change using  $Z$  as the density process, and under the new measure  $\mathsf{Q}$  the process  $Y$  will satisfy the driftless equation  $dY_t = \sigma_0 Y_t^\beta dB_t$  (with a  $\mathsf{Q}$ -Brownian motion  $B$ ), i.e.  $Y$  will be a classical CEV process under  $\mathsf{Q}$ .

Below we apply the results of Section 2 to study the martingale property of  $Z$ . We have  $\mu(x) = \mu_0 x^\alpha$ ,  $\sigma(x) = \sigma_0 x^\beta$ , and  $b(x) = -\mu_0 x^{\alpha-\beta}/\sigma_0$ . Let us note that the case  $\mu_0 = 0$  is trivial and therefore excluded in (29). The auxiliary diffusion  $\tilde{Y}$  of (15) satisfies the driftless SDE  $d\tilde{Y}_t = \sigma_0 \tilde{Y}_t^\beta d\tilde{W}_t$ . Then it follows that  $\tilde{\rho}(x) = 1$  and  $\tilde{s}(x) = x$  for all  $x \in J$ .<sup>3</sup> Since  $\tilde{s}(\infty) = \infty$

<sup>2</sup>To be more precise,  $\tilde{s}(-\infty) = -\infty$  if and only if  $\alpha \geq 1$ .

<sup>3</sup>Note that conditions (20)–(27) are not affected if we add a constant to  $\tilde{s}$ .

the process  $\tilde{Y}$  does not explode at  $\infty$  and, by (25), the boundary point  $\infty$  is bad. At 0 we have  $\tilde{s}(0) > -\infty$  and hence, by (23), the process  $\tilde{Y}$  explodes at 0 if and only if  $\beta < 1$ . Similarly, by (27), we find that 0 is a good boundary point if and only if  $2\beta - \alpha < 1$ .

Theorem 2.1 yields that the process  $Z$  is a strict local martingale if and only if 0 is a bad point and the process  $\tilde{Y}$  explodes at 0. This is equivalent to the conditions  $2\beta - \alpha \geq 1$  and  $\beta < 1$ . Theorem 2.3 implies that  $Z$  is a uniformly integrable martingale if and only if 0 is a good boundary point, in other words when the inequality  $2\beta - \alpha < 1$  holds. This analysis is graphically depicted in Figure 1.

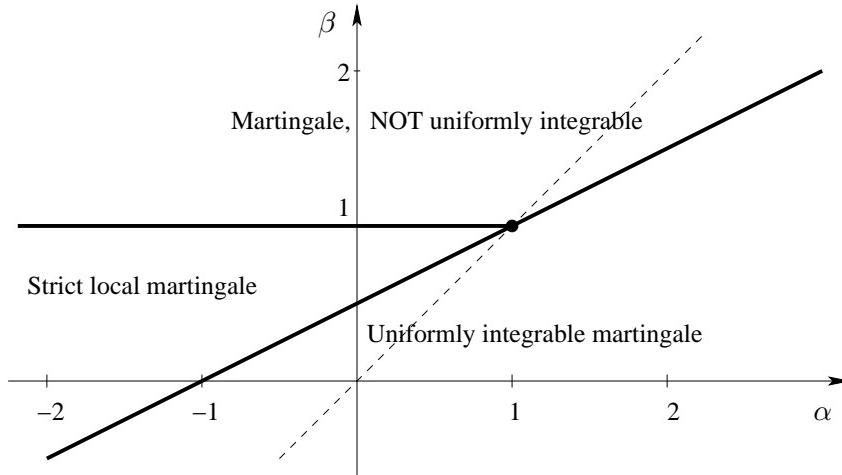


FIGURE 1. The local martingale  $Z$  defined in (30) is a uniformly integrable martingale for all points in the open half-plane under the line  $2\beta - \alpha = 1$ , a strict local martingale in the half-open wedge defined by the inequalities  $\beta < 1$  and  $2\beta - \alpha \geq 1$ , and a martingale that is not uniformly integrable in the closed wedge defined by the inequalities  $\beta \geq 1$  and  $2\beta - \alpha \geq 1$ . Note that, excluding the trivial case  $\mu_0 = 0$ , the parameters  $\mu_0$  and  $\sigma_0$  do not have any bearing on the martingale property of the process  $Z$ . The dot at  $(1, 1)$  corresponds to geometric Brownian motion and the dashed line to the models with  $\alpha = \beta$ .

In the case  $\alpha = \beta$  the stochastic exponential  $Z$  defined in (30) is a geometric Brownian motion stopped at the explosion time  $\zeta$  of the diffusion  $Y$ . As we can see in Figure 1,  $Z$  is a uniformly integrable martingale for  $\alpha = \beta < 1$  and a martingale that is not uniformly integrable for  $\alpha = \beta \geq 1$ . In particular the case  $\alpha = \beta = 0$  implies that the law of Brownian motion with drift stopped at 0 is absolutely continuous with respect to the law of Brownian motion stopped at 0, which is not the case if both processes are allowed to diffuse on the entire real line.

#### 4. FINANCIAL BUBBLES

Bubbles have recently attracted attention in the mathematical finance literature; see e.g. Cox and Hobson [8], Ekström and Tysk [12], Heston, Loewenstein, and Willard [18], Jarrow, Protter, and Shimbo [21], Madan and Yor [30], Pal and Protter [33]. The reason is that option pricing in models with bubbles is a very delicate task. Therefore it is important to have tools for ascertaining the existence or absence of bubbles in financial models. The problem of finding conditions that guarantee the absence of bubbles in certain stochastic volatility models was

studied in Sin [36], Jourdain [22], Andersen and Piterbarg [1]; see also the discussion in Lewis [28, Ch. 9]. In this section we show how to apply Theorems 2.1 and 2.3 to get deterministic criteria for the absence of bubbles in time-homogeneous local volatility models.

Let the discounted price of an asset be modelled by a nonnegative process  $S$ . Under a risk-neutral measure  $\mathsf{P}$  the discounted price is a local martingale. In the terminology of [21] there is a *type 3 bubble* (resp. *type 2 bubble*) in the model if  $S$  is a strict local martingale (resp. a martingale but not a uniformly integrable martingale) under  $\mathsf{P}$ .

The setting in this section is as follows. Let  $J = (0, \infty)$  and  $Y$  be a  $J$ -valued diffusion governed under a measure  $\mathsf{P}$  by the SDE

$$(31) \quad dY_t = \mu_0 Y_t dt + \sigma(Y_t) dW_t, \quad Y_0 = x_0 > 0$$

with  $\mu_0 \in \mathbb{R}$  and a Borel function  $\sigma: J \rightarrow \mathbb{R}$  satisfying the Engelbert–Schmidt conditions (as earlier  $Y$  is stopped after the explosion time  $\zeta$ ). We interpret  $Y$  as a non-discounted asset price with evolution (31) under a risk-neutral measure. Then  $\mu_0$  is the risk-free interest rate<sup>4</sup>, and the discounted price is  $S_t = e^{-\mu_0 t} Y_t$  (note that we also have  $S_t = e^{-\mu_0(t \wedge \zeta)} Y_t$  because  $J = (0, \infty)$ ).

Since  $Y$  is a price process, it is quite natural to assume that  $Y$  does not explode at  $\infty$ . But this holds automatically for (31) as shown in the following lemma. (This result is well-known for  $\sigma$  having linear growth, but it holds also in general. Surprisingly, we could not find this result in standard textbooks. Let us also note that it is not at all easy to get this from Feller's test for explosions.)

**Lemma 4.1.**  *$Y$  does not explode at  $\infty$ .*<sup>5</sup>

*Proof.* Until the explosion time  $\zeta$  we get by Itô's formula

$$e^{-\mu_0 t} Y_t = x_0 + \int_0^t e^{-\mu_0 s} \sigma(Y_s) dW_s, \quad t \in [0, \zeta).$$

By the Dambis–Dubins–Schwarz theorem for continuous local martingales on a stochastic interval (see [35, Ch. V, Ex. (1.18)]), the continuous local martingale  $(e^{-\mu_0 t} Y_t)_{t \in [0, \zeta]}$  is a time-changed Brownian motion, hence  $\mathsf{P}(\zeta < \infty, \lim_{t \uparrow \zeta} e^{-\mu_0 t} Y_t = \infty) = 0$ . This concludes the proof.  $\square$

**Proposition 4.2.** *The discounted price process  $S$  introduced above coincides with the process  $x_0 Z$ , where  $Z$  is given by (7) with  $b(x) = \sigma(x)/x$ .*

The desired criteria for the absence of type 2 and type 3 bubbles now follow immediately from Proposition 4.2 and Theorems 2.1 and 2.3. We omit the formulations.

*Remark.* Note that in this setting formula (17) simplifies to  $\tilde{\rho}(x) = \rho(x)/x^2$ ,  $x \in J$ , since conditions (20)–(27) are not affected if  $\tilde{\rho}$  is multiplied by a positive constant.

*Proof.* Until the explosion time  $\zeta$  we set

$$V_t := \int_0^t \frac{dS_u}{S_u} = \int_0^t \frac{\sigma(Y_u)}{Y_u} dW_u, \quad t \in [0, \zeta).$$

---

<sup>4</sup>Only the case  $\mu_0 \geq 0$  has a financial meaning, but the results below hold for any  $\mu_0 \in \mathbb{R}$ .

<sup>5</sup>However,  $Y$  can explode at 0.

The stochastic integral is well-defined because  $Y$  is strictly positive on  $[0, \zeta)$ . Then we have

$$S_t = x_0 \mathcal{E}(V)_t = x_0 Z_t, \quad t \in [0, \zeta).$$

Both  $S$  and  $Z$  have a limit as  $t \uparrow \zeta$  and are stopped after  $\zeta$ . This completes the proof.  $\square$

In the important case of zero interest rate ( $\mu_0 = 0$ ) the problem under consideration amounts to the question of whether the solution of the now driftless SDE in (31) is a true martingale or a strict local martingale. This problem is of interest in itself, and the answer looks particularly simple. In what follows we will use, for a function  $f: J \rightarrow \mathbb{R}$  and a class of functions  $\mathfrak{M}$ , the notation  $f(x) \in \mathfrak{M}$  as a synonym for  $f \in \mathfrak{M}$ .

**Corollary 4.3.** *Let  $Y = (Y_t)_{t \in [0, \infty)}$  be a solution of the SDE  $dY_t = \sigma(Y_t) dW_t$ ,  $Y_0 = x_0 > 0$ , where  $\sigma: (0, \infty) \rightarrow \mathbb{R}$  is a Borel function satisfying the Engelbert–Schmidt conditions ( $Y$  is stopped after it hits zero).*

- (i) *If  $x/\sigma^2(x) \in L^1_{\text{loc}}(\infty-)$ , then  $Y$  is a strict local martingale.*
- (ii) *If  $x/\sigma^2(x) \notin L^1_{\text{loc}}(\infty-)$ , then  $Y$  is a martingale but is not uniformly integrable.*

*Remark.* Moreover, if  $x/\sigma^2(x) \in L^1_{\text{loc}}(\infty-)$ , then  $Y$  is a strict local martingale on each time interval  $[0, T]$ ,  $T \in (0, \infty)$ .

*Proof.* We can take  $\rho \equiv 1$ ,  $s(x) = x$ ,  $\tilde{s}(x) = \frac{1}{x^2}$ ,  $\tilde{s}(x) = -\frac{1}{x}$ . Since  $s(\infty) = \infty$  and  $\tilde{s}(0) = -\infty$ , neither 0 nor  $\infty$  are good (see (24) and (27)), the process  $Y$  does not explode at  $\infty$  and the diffusion  $\tilde{Y}$  does not explode at 0 (see (21) and (23)). Theorem 2.3 combined with Proposition 4.2 yields that  $Y$  cannot be a uniformly integrable martingale. Theorem 2.1 implies that  $Y$  is a martingale if and only if  $\tilde{Y}$  does not explode at  $\infty$ . The application of (21) completes the proof. Finally, the remark after Corollary 4.3 follows from the remark after Theorem 2.1.  $\square$

The result established in Corollary 4.3 was first proved in Delbaen and Shirakawa [11] under the stronger assumption that the functions  $\sigma$  and  $1/\sigma$  are locally bounded on  $(0, \infty)$ . Later it was proved by a different method in Carr, Cherny, and Urusov [3] without this assumption. Furthermore, Corollary 4.3 can be obtained as a consequence of Theorem 1 in Kotani [26] and of Theorem 3.20 in Hulley and Platen [19]. Let us note that the method of this paper differs from all these approaches.

Let us further mention that it is only the possibility of bubbles in local volatility models that prevents the main theorem in Mijatović [31] from being applied for the pricing of time-dependent barrier options. In particular, the above results could be used to characterise the class of local volatility models within which time-dependent barrier options can be priced by solving a related system of Volterra integral equations (see Section 2 in [31]).

*Remarks.* (i) Let us discuss the question of whether all of the following possibilities can be realised for the process  $S_t = e^{-\mu_0 t} Y_t$ , where  $Y$  is given by (31):

- (1)  $S$  is a strict local martingale;
- (2)  $S$  is a martingale but is not uniformly integrable;
- (3)  $S$  is a uniformly integrable martingale.

Corollary 4.3 implies that the answer is affirmative for (1) and (2) even within the subclass  $\mu_0 = 0$ . Possibility (3) can also be realised: one can take, for example,  $\sigma(x) = \sqrt{x}$  and any  $\mu_0 > 0$  (computations are omitted).

(ii) We now generalise Proposition 4.2 to the case of a stochastic risk-free interest rate. In the setting of Section 2 let us assume that  $J = (l, r)$  with  $0 \leq l < r \leq \infty$ , so that the process  $Y$  governed by (4) is nonnegative. Define the discounted price process until the explosion time  $\zeta$  by the formula

$$S_t = \exp \left\{ - \int_0^t \frac{\mu(Y_u)}{Y_u} du \right\} Y_t, \quad t \in [0, \zeta).$$

By Itô's formula,  $(S_t)_{t \in [0, \zeta]}$  is a nonnegative local martingale on the stochastic interval  $[0, \zeta]$ , hence the limit  $S_\zeta := \lim_{t \uparrow \zeta} S_t$  exists and is finite  $\mathbb{P}$ -a.s. (even if  $r = \infty$  and  $Y$  explodes at  $\infty$ ). Now we define a nonnegative local martingale  $S = (S_t)_{t \in [0, \infty)}$  by stopping  $(S_t)_{t \in [0, \zeta]}$  after  $\zeta$  and interpret it as a discounted price process. In the case of a constant interest rate  $\mu_0$  (i.e.  $\mu(x) = \mu_0 x$ ) and  $J = (0, \infty)$  we get the setting above. Proposition 4.2 holds without any change in this more general setting and is proved in the same way.

**Example 4.4** (Bubbles in the CEV Model). In the setting of this section let  $\sigma(x) = \sigma_0 x^\alpha$ ,  $\sigma_0 > 0$ ,  $\alpha \in \mathbb{R}$ . Then the process  $Y$  given by (31) is called the *constant elasticity of variance (CEV) process* (with drift  $\mu_0$ ). This model was first proposed by Cox [9] for  $\alpha \leq 1$  and Emanuel and MacBeth [14] for  $\alpha > 1$  (see also Davydov and Linetsky [10] and the references therein). It is known that  $S_t = e^{-\mu_0 t} Y_t$  is a true martingale for  $\alpha \leq 1$  and a strict local martingale for  $\alpha > 1$ . This was first proved in [14] by testing the equality  $\mathbb{E} S_t = x_0$  (the marginal densities of  $Y$  are known explicitly in this case). Below we prove this by our method and further investigate when  $S$  is a uniformly integrable martingale.

Let us prove the following claims.

- (i)  $S$  is a martingale if and only if  $\alpha \leq 1$ ;
- (ii)  $S$  is a uniformly integrable martingale if and only if  $\alpha < 1$  and  $\mu_0 > 0$ .

The case  $\alpha = 1$  is clear. We can therefore assume without loss of generality that  $\alpha \neq 1$ .

By Proposition 4.2 we need to study the martingale property of the process  $Z$  defined in (7) with  $b(x) := \sigma_0 x^{\alpha-1}$ ,  $x \in J := (0, \infty)$ . The auxiliary diffusion in (15) takes the form

$$d\tilde{Y}_t = (\mu_0 \tilde{Y}_t + \sigma_0^2 \tilde{Y}_t^{2\alpha-1}) dt + \sigma_0 \tilde{Y}_t^\alpha d\tilde{W}_t, \quad \tilde{Y}_t = x_0.$$

By Feller's test,  $\tilde{Y}$  explodes at  $\infty$  if and only if  $\alpha > 1$ . It follows from the important remark preceding Theorem 2.1 that the endpoint  $\infty$  is bad whenever  $\tilde{Y}$  explodes at  $\infty$  (recall that  $Y$  does not explode at  $\infty$ ). By Theorem 2.1,  $S$  is a strict local martingale for  $\alpha > 1$ .

It remains to consider the case  $\alpha < 1$ . As was noted above,  $\tilde{Y}$  does not explode at  $\infty$  in this case. A direct computation shows that  $\tilde{s}(0) = -\infty$ . In particular,  $\tilde{Y}$  does not explode at 0. By Theorem 2.1  $S$  is a true martingale for  $\alpha < 1$ . Further, by Theorem 2.3,  $S$  is a uniformly integrable martingale if and only if  $\infty$  is good. Verifying (24) (in the case  $\alpha < 1$ ) we get that  $\infty$  is good if and only if  $\mu_0 > 0$ . This completes the proof of the claims (i) and (ii).

It is interesting to note that the value of  $\mu_0$  plays a role in the classification (i)–(ii) above. Contrary to the case where  $\mu_0 = 0$  (the CEV model without drift), the possibility that  $S$  is a uniformly integrable martingale can be realised with positive interest rates. In other words, taking  $\mu_0 > 0$  may remove a type 2 bubble in this model, but has no effect on the existence of a type 3 bubble.

## 5. SEPARATING TIMES

In order to present the proofs of our main results we need the concept of a *separating time* for a pair of measures on a filtered space, which was introduced in Cherny and Urusov [6]. In this section we recall the definition of separating times and describe the explicit form of separating times for distributions of solutions of SDEs. All results stated in this section are taken from Cherny and Urusov [7] (see also [6]).

**5.1. Definition of Separating Times.** Let  $(\Omega, \mathcal{F})$  be a measurable space endowed with a right-continuous filtration  $(\mathcal{F}_t)_{t \in [0, \infty)}$ . We recall that the  $\sigma$ -field  $\mathcal{F}_\tau$ , for any  $(\mathcal{F}_t)$ -stopping time  $\tau$ , is defined by

$$(32) \quad \mathcal{F}_\tau = \{A \in \mathcal{F}: A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \in [0, \infty)\}.^6$$

In particular,  $\mathcal{F}_\infty = \mathcal{F}$  by this definition. Note that we do not assume here that  $\mathcal{F} = \bigvee_{t \in [0, \infty)} \mathcal{F}_t$ . Let  $P$  and  $\tilde{P}$  be probability measures on  $(\Omega, \mathcal{F})$ . As usual,  $P_\tau$  (resp.  $\tilde{P}_\tau$ ) denotes the restriction of  $P$  (resp.  $\tilde{P}$ ) to  $(\Omega, \mathcal{F}_\tau)$ . In what follows, it will be convenient for us to consider the extended positive half-line  $[0, \infty] \cup \{\delta\}$ , where  $\delta$  is an additional point. We order  $[0, \infty] \cup \{\delta\}$  in the following way: we take the usual order on  $[0, \infty]$  and let  $\infty < \delta$ .

**Definition 5.1.** An *extended stopping time* is a map  $\tau: \Omega \rightarrow [0, \infty] \cup \{\delta\}$  such that  $\{\tau \leq t\} \in \mathcal{F}_t$  for any  $t \in [0, \infty]$ .

In order to introduce the notion of a separating time, we need to formulate the following result.

**Proposition 5.2.** (i) *There exists an extended stopping time  $S$  such that, for any stopping time  $\tau$ ,*

$$(33) \quad \tilde{P}_\tau \sim P_\tau \text{ on the set } \{\tau < S\},$$

$$(34) \quad \tilde{P}_\tau \perp P_\tau \text{ on the set } \{\tau \geq S\}.$$

(ii) *If  $S'$  is another extended stopping time with these properties, then  $S' = S$   $P, \tilde{P}$ -a.s.*

**Definition 5.3.** A *separating time* for  $P$  and  $\tilde{P}$  (or, more precisely, for  $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, \tilde{P})$ ) is an extended stopping time  $S$  that satisfies (33) and (34) for all stopping times  $\tau$ .

*Remark.* We stress that the separating time for  $P$  and  $\tilde{P}$  is determined  $P, \tilde{P}$ -a.s. uniquely.

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<sup>6</sup>Definition (32) is used e.g. in Jacod and Shiryaev [20]. We draw the reader's attention to the fact that there is also an alternative definition of  $\mathcal{F}_\tau$  in the literature (see e.g. Revuz and Yor [35]), which is obtained by intersecting the  $\sigma$ -field in (32) with the  $\sigma$ -field  $\bigvee_{t \in [0, \infty)} \mathcal{F}_t$ .

Informally, Proposition 5.2 states that two measures  $\mathsf{P}$  and  $\tilde{\mathsf{P}}$  are equivalent up to a random time  $S$  and become singular after that. The equality  $S = \delta$  means that  $\mathsf{P}$  and  $\tilde{\mathsf{P}}$  never become singular, i.e. they are equivalent at time infinity. On the contrary,  $S = \infty$  means that  $\mathsf{P}$  and  $\tilde{\mathsf{P}}$  are equivalent at finite times and become singular at time infinity.

Let us now provide a statistical interpretation of separating times, which yields an intuitive understanding of the notion. Suppose that we deal with the problem of sequentially distinguishing between two statistical hypotheses  $\mathsf{P}$  and  $\tilde{\mathsf{P}}$ , where the information available to us at time  $t$  is described by the  $\sigma$ -field  $\mathcal{F}_t$ . (In particular, if the filtration  $(\mathcal{F}_t)_{t \in [0, \infty)}$  is the natural filtration of some process  $X = (X_t)_{t \in [0, \infty)}$ , we observe sequentially a path of  $X$ .) Then, before the separating time  $S$  occurs, we cannot know with certainty what the true hypothesis is; and as soon as  $S$  occurs we can determine the true hypothesis with certainty.

In fact, the knowledge of the separating time for  $\mathsf{P}$  and  $\tilde{\mathsf{P}}$  yields the knowledge of the mutual arrangement of  $\mathsf{P}$  and  $\tilde{\mathsf{P}}$  from the viewpoint of their absolute continuity and singularity. This is illustrated by the following result. As usual,  $\tilde{\mathsf{P}} \ll \mathsf{P}$  (resp.  $\tilde{\mathsf{P}} \stackrel{\text{loc}}{\sim} \mathsf{P}$ ) means that  $\tilde{\mathsf{P}}_t \ll \mathsf{P}_t$  (resp.  $\tilde{\mathsf{P}}_t \sim \mathsf{P}_t$ ) for all  $t \in [0, \infty)$ .

**Lemma 5.4.** *Let  $S$  be a separating time for  $\mathsf{P}$  and  $\tilde{\mathsf{P}}$ . Then*

- (i)  $\tilde{\mathsf{P}} \sim \mathsf{P} \iff S = \delta \ \tilde{\mathsf{P}}\text{-a.s.};$
- (ii)  $\tilde{\mathsf{P}} \ll \mathsf{P} \iff S = \delta \ \tilde{\mathsf{P}}\text{-a.s.};$
- (iii)  $\tilde{\mathsf{P}} \stackrel{\text{loc}}{\sim} \mathsf{P} \iff S \geq \infty \ \mathsf{P}, \tilde{\mathsf{P}}\text{-a.s.};$
- (iv)  $\tilde{\mathsf{P}} \stackrel{\text{loc}}{\ll} \mathsf{P} \iff S \geq \infty \ \tilde{\mathsf{P}}\text{-a.s.};$
- (v)  $\tilde{\mathsf{P}} \perp \mathsf{P} \iff S \leq \infty \ \mathsf{P}, \tilde{\mathsf{P}}\text{-a.s.} \iff S \leq \infty \ \mathsf{P}\text{-a.s.}$
- (vi)  $\tilde{\mathsf{P}}_0 \perp \mathsf{P}_0 \iff S = 0 \ \mathsf{P}, \tilde{\mathsf{P}}\text{-a.s.} \iff S = 0 \ \mathsf{P}\text{-a.s.}$

*Remark.* Other types of the mutual arrangement of  $\mathsf{P}$  and  $\tilde{\mathsf{P}}$  are also easily expressed in terms of the separating time. For example, we have

$$\begin{aligned} \tilde{\mathsf{P}}_t \ll \mathsf{P}_t &\iff S > t \ \tilde{\mathsf{P}}\text{-a.s.}, \\ \tilde{\mathsf{P}}_t \perp \mathsf{P}_t &\iff S \leq t \ \mathsf{P}, \tilde{\mathsf{P}}\text{-a.s.} \iff S \leq t \ \mathsf{P}\text{-a.s.} \end{aligned}$$

for any  $t \in [0, \infty]$ .

The proof of Lemma 5.4 and the remark after it is straightforward.

**5.2. Separating Times for SDEs.** Let us consider the state space  $J = (l, r)$ ,  $-\infty \leq l < r \leq \infty$ , and set  $\bar{J} = [l, r]$ . In this subsection we use the notation  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)})$  for a certain canonical filtered space and  $\mathsf{P}$  and  $\tilde{\mathsf{P}}$  will be measures on this space, namely, distributions of solutions of SDEs. More precisely, let  $\Omega := \overline{C}([0, \infty), J)$  be the space of continuous functions  $\omega: [0, \infty) \rightarrow \bar{J}$  that start inside  $J$  and can explode, i.e. there exists  $\zeta(\omega) \in (0, \infty]$  such that  $\omega(t) \in J$  for  $t < \zeta(\omega)$  and in the case  $\zeta(\omega) < \infty$  we have either  $\omega(t) = r$  for  $t \geq \zeta(\omega)$  (hence, also  $\lim_{t \uparrow \zeta(\omega)} \omega(t) = r$ ) or  $\omega(t) = l$  for  $t \geq \zeta(\omega)$  (hence, also  $\lim_{t \uparrow \zeta(\omega)} \omega(t) = l$ ). We denote the coordinate process on  $\Omega$  by  $X$  and consider the right-continuous canonical filtration  $\mathcal{F}_t = \bigcap_{\varepsilon > 0} \sigma(X_s: s \in [0, t + \varepsilon])$  and the  $\sigma$ -field  $\mathcal{F} = \bigvee_{t \in [0, \infty)} \mathcal{F}_t$ . Note that the random variable

$\zeta$  described above is the explosion time of  $X$ . Let the measures  $\mathsf{P}$  and  $\tilde{\mathsf{P}}$  on  $(\Omega, \mathcal{F})$  be the distributions of (unique in law) solutions of the SDEs

$$(35) \quad dY_t = \mu(Y_t) dt + \sigma(Y_t) dW_t, \quad Y_0 = x_0,$$

$$(36) \quad d\tilde{Y}_t = \tilde{\mu}(\tilde{Y}_t) dt + \tilde{\sigma}(\tilde{Y}_t) d\tilde{W}_t, \quad \tilde{Y}_0 = x_0,$$

where the coefficients  $\mu, \sigma$  as well as  $\tilde{\mu}, \tilde{\sigma}$  are Borel functions  $J \rightarrow \mathbb{R}$  satisfying the Engelbert–Schmidt conditions (5)–(6).

To summarize the setting, our input consists of the functions  $\mu, \sigma, \tilde{\mu}$ , and  $\tilde{\sigma}$ . Given these functions we get the measures  $\mathsf{P}$  and  $\tilde{\mathsf{P}}$  on our canonical space. As an output we will present an explicit expression for the separating time  $S$  for  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P}, \tilde{\mathsf{P}})$ .

Let  $s$  denote the scale function of diffusion (35) and  $\rho$  the derivative of  $s$  (see (16) and (18)). Similarly, let  $\tilde{s}$  be the scale function of (36) and  $\tilde{\rho}$  its derivative. By  $L_{\text{loc}}^1(x)$  with  $x \in J$  we denote the set of Borel functions that are integrable in a sufficiently small neighbourhood of  $x$ . Similarly we introduce the notation  $L_{\text{loc}}^1(r-)$  and  $L_{\text{loc}}^1(l+)$ . Let  $\nu_L$  denote the Lebesgue measure on  $J$ .

We say that a point  $x \in J$  is *non-separating* if  $\sigma^2 = \tilde{\sigma}^2$   $\nu_L$ -a.e. in a sufficiently small neighbourhood of  $x$  and  $(\mu - \tilde{\mu})^2/\sigma^4 \in L_{\text{loc}}^1(x)$ . We say that the right endpoint  $r$  of  $J$  is *non-separating* if all the points from  $[x_0, r)$  are non-separating as well as

$$(37) \quad s(r) < \infty \text{ and } (s(r) - s) \frac{(\mu - \tilde{\mu})^2}{\rho \sigma^4} \in L_{\text{loc}}^1(r-).$$

We say that the left endpoint  $l$  of  $J$  is *non-separating* if all the points from  $(l, x_0]$  are non-separating as well as

$$(38) \quad s(l) > -\infty \text{ and } (s - s(l)) \frac{(\mu - \tilde{\mu})^2}{\rho \sigma^4} \in L_{\text{loc}}^1(l+).$$

A point in  $\overline{J}$  that is not non-separating is called *separating*. Let  $D$  denote the set of separating points in  $\overline{J}$ . Clearly,  $D$  is closed in  $\overline{J}$ . Let us define

$$D^\varepsilon = \{x \in \overline{J}: \rho(x, D) < \varepsilon\}$$

with the convention  $\emptyset^\varepsilon := \emptyset$ , where  $\rho(x, y) = |\arctan x - \arctan y|$ ,  $x, y \in \overline{J}$  (the metric  $\rho$  induces the standard topology on  $\overline{J}$ ; we could not use here the standard Euclidean metric  $|x - y|$ , as  $\overline{J}$  can contain  $\infty$  or  $-\infty$ ).

**Theorem 5.5.** *Let the functions  $\mu, \sigma$  and  $\tilde{\mu}, \tilde{\sigma}$  satisfy conditions (5)–(6). Let the measures  $\mathsf{P}$  and  $\tilde{\mathsf{P}}$  on the canonical space  $(\Omega, \mathcal{F})$  be the distributions of solutions of SDEs (35) and (36). Then the separating time  $S$  for  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P}, \tilde{\mathsf{P}})$  has the following form.*

- (i) *If  $\mathsf{P} = \tilde{\mathsf{P}}$ , then  $S = \delta$   $\mathsf{P}, \tilde{\mathsf{P}}$ -a.s.*
- (ii) *If  $\mathsf{P} \neq \tilde{\mathsf{P}}$ , then*

$$S = \sup_n \overline{\inf}\{t \in [0, \infty): X_t \in D^{1/n}\} \quad \mathsf{P}, \tilde{\mathsf{P}}$$
-a.s.,

where “ $\overline{\inf}$ ” is the same as “ $\inf$ ” except that  $\overline{\inf} \emptyset := \delta$ .

*Remarks.* (i) Theorem 5.5 is proved in [7] in the particular case where  $J = (-\infty, \infty)$  (see Theorem 5.7 in [7]). The case of general  $J = (l, r)$  can be reduced to the case  $J = (-\infty, \infty)$  by considering a diffeomorphism  $(l, r) \rightarrow (-\infty, \infty)$ . We omit the tedious but straightforward computations.

(ii) Let us explain the structure of  $S$  in the case  $\mathsf{P} \neq \tilde{\mathsf{P}}$ . Denote by  $\alpha$  the “separating point that is closest to  $x_0$  from the left-hand side”, i.e.

$$\alpha = \begin{cases} \sup\{x : x \in [l, x_0] \cap D\} & \text{if } [l, x_0] \cap D \neq \emptyset, \\ \Delta & \text{if } [l, x_0] \cap D = \emptyset, \end{cases}$$

where  $\Delta$  is an additional point ( $\Delta \notin \overline{J}$ ). Let us consider the “hitting time of  $\alpha$ ”:

$$U = \begin{cases} \delta & \text{if } \alpha = \Delta, \\ \delta & \text{if } \alpha = l \text{ and } \liminf_{t \uparrow \zeta} X_t > l, \\ \zeta & \text{if } \alpha = l \text{ and } \liminf_{t \uparrow \zeta} X_t = l, \\ \bar{\tau}_\alpha & \text{if } \alpha > l, \end{cases}$$

where  $\bar{\tau}_\alpha = \overline{\inf}\{t \in [0, \infty) : X_t = \alpha\}$ . Similarly, let us denote by  $\beta$  the “separating point that is closest to  $x_0$  from the right-hand side” and by  $V$  the “hitting time of  $\beta$ ”. Then  $S = U \wedge V$   $\mathsf{P}, \tilde{\mathsf{P}}$ -a.s. This follows from Proposition A.2.

(iii) We need the following statements proved in remark (ii) after Theorem 5.7 in [7]. If  $[x_0, r) \subseteq \overline{J} \setminus D$ , then condition (37) is equivalent to

$$(39) \quad \tilde{s}(r) < \infty \text{ and } (\tilde{s}(r) - \tilde{s}) \frac{(\mu - \tilde{\mu})^2}{\tilde{\rho} \tilde{\sigma}^4} \in L^1_{\text{loc}}(r-).$$

If  $(l, x_0] \subseteq \overline{J} \setminus D$ , then condition (38) is equivalent to

$$(40) \quad \tilde{s}(l) > -\infty \text{ and } (\tilde{s} - \tilde{s}(l)) \frac{(\mu - \tilde{\mu})^2}{\tilde{\rho} \tilde{\sigma}^4} \in L^1_{\text{loc}}(l+).$$

In the context of Section 2 SDE (36) has a particular form (15), i.e. we have  $\tilde{\sigma} = \sigma$  and  $\tilde{\mu} = \mu + b\sigma$ . Condition (8) ensures that  $J \subseteq \overline{J} \setminus D$ . Conditions (37), (39), (38), and (40) reduce respectively to conditions (24), (25), (26), and (27). We get the equivalence between (24) and (25) as well as between (26) and (27).

(iv) Let us show that the endpoint  $r$  (resp.  $l$ ) is separating whenever one of the diffusions (35) and (36) explodes at  $r$  (resp. at  $l$ ) and the other does not. Suppose that  $r$  is non-separating. Since the set of non-separating points is open in  $\overline{J}$ , there exists  $a < x_0$  such that all the points in  $(a, r]$  are non-separating. Set

$$E = \left\{ \lim_{t \uparrow \zeta} X_t = r, X_t > a \ \forall t \in [0, \zeta) \right\}$$

and note that  $\mathsf{P}(E) > 0$  by Proposition A.3. By Theorem 5.5, we have  $S = \delta$  on  $E$ , hence  $\tilde{\mathsf{P}} \sim \mathsf{P}$  on  $E$ . If  $X$  explodes at  $r$  under  $\mathsf{P}$ , then, by Proposition A.3,  $\mathsf{P}(E \cap \{\zeta < \infty\}) > 0$ , hence  $\tilde{\mathsf{P}}(E \cap \{\zeta < \infty\}) > 0$ , i.e.  $X$  explodes at  $r$  also under  $\tilde{\mathsf{P}}$ . Similarly, if  $X$  does not explode at  $r$  under  $\mathsf{P}$ , then it does not explode at  $r$  also under  $\tilde{\mathsf{P}}$ .

In the context of Section 2 (i.e. when  $\tilde{\sigma} = \sigma$  and  $\tilde{\mu} = \mu + b\sigma$ ) we have that the endpoint  $r$  is good (see (24)) if and only if it is non-separating (see (37) and note that all points in  $J$  are non-separating due to (8)). Thus,  $r$  is bad whenever one of the diffusions (4) and (15) explodes at  $r$  and the other does not. This proves the important remark preceding Theorem 2.1.

## 6. PROOFS OF THEOREMS 2.1 AND 2.3

In this section we prove the results formulated in Section 2. We fix some number  $T \in (0, \infty)$  and consider the following questions for the process  $Z$  defined in (7):

- (i) Is the process  $(Z_t)_{t \in [0, T]}$  a martingale?
- (ii) Is the process  $(Z_t)_{t \in [0, \infty)}$  a uniformly integrable martingale?

The initial step is to translate these questions for the process  $Z$  of (7) defined on an arbitrary filtered probability space into the related questions on the canonical filtered space. In contrast to Section 2, we use throughout this section the notation  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)})$  for the canonical filtered space defined in Section 5.2. By  $X$  we denote the coordinate process on  $\Omega$  and by  $\zeta$  the explosion time of  $X$ . Let the probability  $\mathsf{P}$  on  $(\Omega, \mathcal{F})$  be the distribution of a (unique in law) solution of SDE (4).

Let us define a process  $(W_t)_{t \in [0, \zeta)}$  on the stochastic interval  $[0, \zeta)$  by the formula

$$W_t = \int_0^t \frac{1}{\sigma(X_u)} (dX_u - \mu(X_u) du), \quad t < \zeta.$$

Then it is a continuous  $(\mathcal{F}_t, \mathsf{P})$ -local martingale on the stochastic interval  $[0, \zeta)$  with  $\langle W, W \rangle_t = t$ ,  $t < \zeta$ . Hence,  $(W_t)_{t \in [0, \zeta)}$  can be extended to a Brownian motion on the time interval  $[0, \infty)$  on an enlargement of the probability space (see [35, Ch. V, § 1]). Consequently, the process  $(W_t)_{t \in [0, \zeta)}$  has a finite limit  $\mathsf{P}$ -a.s. as  $t \uparrow \zeta$ , and we define the process  $W = (W_t)_{t \in [0, \infty)}$  by stopping  $(W_t)_{t \in [0, \zeta)}$  after  $\zeta$  (and therefore do not enlarge the canonical probability space). Thus,  $W$  is a Brownian motion stopped after  $\zeta$ . Finally, we set

$$(41) \quad Z_t = \exp \left\{ \int_0^{t \wedge \zeta} b(X_u) dW_u - \frac{1}{2} \int_0^{t \wedge \zeta} b^2(X_u) du \right\}, \quad t \in [0, \infty)$$

$(Z_t := 0 \text{ for } t \geq \zeta \text{ on } \{\zeta < \infty, \int_0^\zeta b^2(X_u) du = \infty\})$ . The thus defined process  $Z$  on  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$  is a nonnegative  $(\mathcal{F}_t, \mathsf{P})$ -local martingale. Now the answer to the question (i) (resp. (ii)) for the process in (7) is positive if and only if the answer to the question (i) (resp. (ii)) for  $Z$  of (41) is positive.<sup>7</sup> Thus, below we will consider the questions (i)–(ii) for the process  $Z$  of (41). This reduces the initial problems to the problems in the canonical setting.

Let  $(a_n)_{n \in \mathbb{N}}$  and  $(c_n)_{n \in \mathbb{N}}$  be strictly monotone sequences such that  $a_1 < x_0 < c_1$ ,  $a_n \downarrow l$ , and  $c_n \uparrow r$ . We set

$$\tau_n = \inf \{t \in [0, \infty) : X_t \notin (a_n, c_n)\}$$

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<sup>7</sup>There is a difficulty in that the filtration in Section 2 need not be generated by  $Y$ , while the canonical filtration  $(\mathcal{F}_t)$  in this section is the right-continuous filtration generated by  $X$ . However, this equivalence holds because both in the case of (7) and in the case of (41) the property that  $(Z_t)_{t \in [0, T]}$  is a martingale (resp.  $(Z_t)_{t \in [0, \infty)}$  is a uniformly integrable martingale) is equivalent to the filtration-independent property  $\mathsf{E}Z_T = 1$  (resp.  $\mathsf{E}Z_\infty = 1$ ).

with  $\inf \emptyset := \infty$  and note that  $\tau_n \uparrow \zeta$  for all  $\omega \in \Omega$  and  $\tau_n < \zeta$  on the set  $\{\zeta < \infty\}$ . As in Section 5 we denote by  $\mathcal{F}_\tau$ , for any  $(\mathcal{F}_t)$ -stopping time  $\tau$ , the  $\sigma$ -field defined in (32). Now we need to prove several lemmas.

**Lemma 6.1.** *For any  $t \in [0, \infty]$  we have  $\bigvee_{n \in \mathbb{N}} \mathcal{F}_{t \wedge \tau_n} = \mathcal{F}_t$ .*

*Proof.* We start by proving that

$$(42) \quad \bigvee_{n \in \mathbb{N}} \mathcal{F}_{t \wedge \tau_n} = \mathcal{F} \text{ on the set } \{\zeta \leq t\}$$

for any  $t \in [0, \infty]$ . Indeed, this holds because  $\mathcal{F} = \sigma(X_s : s \in [0, \infty))$  and for any  $s \in [0, \infty)$  we have

$$X_s = \lim_{n \rightarrow \infty} X_{t \wedge \tau_n \wedge s} \text{ on } \{\zeta \leq t\}.$$

The latter holds because  $X$  stays after the explosion time  $\zeta$  at that endpoint of  $J$  at which it explodes (in particular,  $X_s = X_{s \wedge \zeta}$ ).

The inclusion  $\bigvee_{n \in \mathbb{N}} \mathcal{F}_{t \wedge \tau_n} \subseteq \mathcal{F}_t$  is clear. For the reverse inclusion take an arbitrary set  $A \in \mathcal{F}_t$  and express it as

$$A = \left[ \bigcup_{n \in \mathbb{N}} (A \cap \{t < \tau_n\}) \right] \cup [A \cap \{\zeta \leq t\}].$$

The set  $A \cap \{t < \tau_n\}$  belongs both to  $\mathcal{F}_t$  and to  $\mathcal{F}_{\tau_n}$ , hence to  $\mathcal{F}_t \cap \mathcal{F}_{\tau_n} = \mathcal{F}_{t \wedge \tau_n}$ . Finally, by (42),  $A \cap \{\zeta \leq t\} \in \bigvee_{n \in \mathbb{N}} \mathcal{F}_{t \wedge \tau_n}$ .  $\square$

**Lemma 6.2.** *For any  $n \in \mathbb{N}$  we have  $\bigvee_{t \in [0, \infty)} \mathcal{F}_{t \wedge \tau_n} = \mathcal{F}_{\tau_n}$ .*

The proof is similar to that of Lemma 6.1.

In what follows the probability measure  $\tilde{P}$  on  $(\Omega, \mathcal{F})$  is the distribution of a (unique in law) solution of (15). As in Section 5, for any  $(\mathcal{F}_t)$ -stopping time  $\tau$ ,  $P_\tau$  (resp.  $\tilde{P}_\tau$ ) denotes the restriction of  $P$  (resp.  $\tilde{P}$ ) to the  $\sigma$ -field  $\mathcal{F}_\tau$ . By  $S$  we denote the separating time for  $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, \tilde{P})$  (see Definition 5.3).

**Lemma 6.3.** *For any  $n \in \mathbb{N}$  we have  $\tilde{P}_{\tau_n} \sim P_{\tau_n}$  and*

$$(43) \quad \frac{d\tilde{P}_{t \wedge \tau_n}}{dP_{t \wedge \tau_n}} = Z_{t \wedge \tau_n} \quad P\text{-a.s.}, \quad t \in [0, \infty].$$

*Proof.* Let us fix  $n \in \mathbb{N}$ . We want to apply Theorem 5.5 to our measures  $P$  and  $\tilde{P}$ . In our case all points inside  $J$  are non-separating due to (8). Hence,

$$S \geq \zeta > \tau_n \quad P, \tilde{P}\text{-a.s.},$$

where the first inequality follows from remark (ii) after Theorem 5.5 and the second one from Proposition A.2. By the definition of a separating time, we get  $\tilde{P}_{\tau_n} \sim P_{\tau_n}$ . It remains to prove (43).

We consider a càdlàg version of the density process  $D_t^{(n)} := \frac{d\tilde{P}_{t \wedge \tau_n}}{dP_{t \wedge \tau_n}}$ ,  $t \in [0, \infty)$ , which is an  $(\mathcal{F}_{t \wedge \tau_n}, P)$ -martingale closed by the  $P$ -a.s. strictly positive random variable  $\frac{d\tilde{P}_{\tau_n}}{dP_{\tau_n}}$ . Then the

processes  $D^{(n)}$  and  $D_{-}^{(n)}$  are  $\mathsf{P}$ -a.s. strictly positive (see [20, Ch. III, Lem. 3.6]). Since  $\frac{d\tilde{\mathsf{P}}_{\tau_n}}{d\mathsf{P}_{\tau_n}}$  is  $\mathcal{F}_{\tau_n}$ -measurable, we have

$$D_t^{(n)} = \mathsf{E}_{\mathsf{P}} \left( \frac{d\tilde{\mathsf{P}}_{\tau_n}}{d\mathsf{P}_{\tau_n}} \middle| \mathcal{F}_{t \wedge \tau_n} \right) = \mathsf{E}_{\mathsf{P}} \left( \frac{d\tilde{\mathsf{P}}_{\tau_n}}{d\mathsf{P}_{\tau_n}} \middle| \mathcal{F}_t \right) \quad \mathsf{P}\text{-a.s.}, \quad t \in [0, \infty).$$

Hence,  $D^{(n)}$  is also an  $(\mathcal{F}_t, \mathsf{P})$ -martingale. We obtain that the stochastic logarithm

$$(44) \quad L_t^{(n)} := \int_0^t \frac{dD_u^{(n)}}{D_{u-}^{(n)}}, \quad t \in [0, \infty)$$

is a well-defined  $(\mathcal{F}_t, \mathsf{P})$ -local martingale stopped after  $\tau_n$  (as  $D^{(n)}$  is stopped after  $\tau_n$ ). Let us prove that

$$(45) \quad L_t^{(n)} = \int_0^{t \wedge \tau_n} b(X_u) dW_u \quad \mathsf{P}\text{-a.s.}, \quad t \in [0, \infty).$$

To prove (45) we first need to argue that there exists a predictable process  $H^{(n)}$  integrable with respect to the  $(\mathcal{F}_t, \mathsf{P})$ -local martingale

$$M_t^{(n)} := X_{t \wedge \tau_n} - \int_0^{t \wedge \tau_n} \mu(X_u) du, \quad t \in [0, \infty)$$

such that

$$(46) \quad L_t^{(n)} = \int_0^t H_u^{(n)} dM_u^{(n)} \quad \mathsf{P}\text{-a.s.}, \quad t \in [0, \infty).$$

Note that existence of such a process  $H^{(n)}$  will imply that both  $L^{(n)}$  and  $D^{(n)}$  are  $\mathsf{P}$ -a.s. continuous processes (as  $D^{(n)}$  is the stochastic exponential of  $L^{(n)}$ ). The idea is to use the Fundamental Representation Theorem (see [20, Ch. III, Th. 4.29]) in order to prove the existence of  $H^{(n)}$ . However, we cannot apply the Fundamental Representation Theorem directly because the process  $X$  can be explosive under  $\mathsf{P}$  and thus may not be a semimartingale.

In order to avoid this problem we consider a probability measure  $\mathsf{P}'$  on the canonical filtered space, which is the distribution of a (unique in law) solution of the SDE

$$(47) \quad dY_t = \mu'(Y_t) dt + \sigma'(Y_t) dB_t, \quad Y_0 = x_0,$$

where  $B$  denotes some Brownian motion. The Borel functions  $\mu', \sigma': J \rightarrow \mathbb{R}$  are given by  $\mu' = \mu$  and  $\sigma' = \sigma$  on  $[a_{n+1}, c_{n+1}]$ . On the complement  $J \setminus [a_{n+1}, c_{n+1}]$  they are chosen so that the Engelbert–Schmidt conditions (5) and (6) are satisfied and so that the coordinate process  $X$  is non-explosive under  $\mathsf{P}'$ . Then by [4, Th. 2.11] we obtain the equality  $\mathsf{P}_{\tau_n} = \mathsf{P}'_{\tau_n}$ . Since the process  $L^{(n)}$  is stopped after  $\tau_n$ , it is  $\mathcal{F}_{\tau_n}$ -measurable. Let  $\eta_m \uparrow \infty$   $\mathsf{P}$ -a.s. be a localizing sequence for  $L^{(n)}$ , i.e.  $(L^{(n)})^{\eta_m}$  are  $(\mathcal{F}_t, \mathsf{P})$ -martingales. If we define  $\eta'_m := \eta_m \wedge \tau_n$  and use the fact  $\mathsf{P}_{\tau_n} = \mathsf{P}'_{\tau_n}$ , we find that  $\eta'_m \uparrow \tau_n$   $\mathsf{P}'$ -a.s. as  $m \uparrow \infty$  and that  $(L^{(n)})^{\eta'_m}$  is an  $(\mathcal{F}_t, \mathsf{P}')$ -martingale. Therefore,  $L^{(n)}$  is an  $(\mathcal{F}_t, \mathsf{P}')$ -local martingale on the stochastic interval  $[0, \tau_n]$  (we cannot guarantee more knowing only  $\mathsf{P}_{\tau_n} = \mathsf{P}'_{\tau_n}$ ; for instance, we cannot guarantee that  $\eta_m \uparrow \infty$   $\mathsf{P}'$ -a.s. because  $\mathsf{P}'$  need not be locally absolutely continuous with respect to  $\mathsf{P}$ ). Now the Fundamental Representation

Theorem applied to all  $(L^{(n)})^{\eta'_m}$ ,  $m \in \mathbb{N}$ , implies the existence of a predictable process  $H'$  on the filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathsf{P}')$  such that

$$(48) \quad L_t^{(n)} = \int_0^t H'_u (dX_u - \mu'(X_u) du) \quad \mathsf{P}'\text{-a.s.}, \quad t \in [0, \tau_n],$$

since uniqueness in law holds for SDE (47). We now obtain (46) by setting  $H_u^{(n)} := H'_u I(u \leq \tau_n)$ , stopping the integrator in (48) after  $\tau_n$ , and using the equality  $\mathsf{P}_{\tau_n} = \mathsf{P}'_{\tau_n}$ .

We get from (44) that

$$(49) \quad \frac{d\tilde{\mathsf{P}}_{t \wedge \tau_n}}{d\mathsf{P}_{t \wedge \tau_n}} = \mathcal{E}(L^{(n)})_t \quad \mathsf{P}\text{-a.s.}, \quad t \in [0, \infty).$$

The process  $M^{(n)}$  is a continuous  $(\mathcal{F}_t, \mathsf{P})$ -local martingale stopped after  $\tau_n$  and is therefore also an  $(\mathcal{F}_{t \wedge \tau_n}, \mathsf{P})$ -local martingale. By Girsanov's theorem for local martingales (see [20, Ch. III, Th. 3.11]) applied on the filtration  $(\mathcal{F}_{t \wedge \tau_n})_{t \in [0, \infty]}$  and by formulas (49) and (46), the process

$$M_t^{(n)} - \langle M^{(n)}, L^{(n)} \rangle_t = X_{t \wedge \tau_n} - \int_0^{t \wedge \tau_n} (\mu(X_u) + H_u^{(n)} \sigma^2(X_u)) du, \quad t \in [0, \infty)$$

is an  $(\mathcal{F}_{t \wedge \tau_n}, \tilde{\mathsf{P}})$ -local martingale. Since  $\tilde{\mathsf{P}}$  is the distribution of a solution of SDE (15), the process

$$X_{t \wedge \tau_n} - \int_0^{t \wedge \tau_n} (\mu + b\sigma)(X_u) du, \quad t \in [0, \infty)$$

is an  $(\mathcal{F}_t, \tilde{\mathsf{P}})$ -local martingale and therefore also an  $(\mathcal{F}_{t \wedge \tau_n}, \tilde{\mathsf{P}})$ -local martingale. The process obtained as a difference of these two  $(\mathcal{F}_{t \wedge \tau_n}, \tilde{\mathsf{P}})$ -local martingales

$$\int_0^{t \wedge \tau_n} (b(X_u) \sigma(X_u) - H_u^{(n)} \sigma^2(X_u)) du, \quad t \in [0, \infty)$$

is a continuous  $(\mathcal{F}_{t \wedge \tau_n}, \tilde{\mathsf{P}})$ -local martingale of finite variation starting from zero. Hence, it is identically zero. Consequently,  $\tilde{\mathsf{P}}$ -a.s. we have

$$(50) \quad H_u^{(n)} = \frac{b}{\sigma}(X_u) \text{ for } \nu_L\text{-a.e. } u \in [0, \tau_n],$$

where  $\nu_L$  denotes the Lebesgue measure. Since  $\tilde{\mathsf{P}}_{\tau_n} \sim \mathsf{P}_{\tau_n}$ , equality (50) holds also  $\mathsf{P}$ -a.s., and (45) follows from (46). Now (43) for  $t \in [0, \infty)$  follows from (49) and (45), and it remains only to prove (43) for  $t = \infty$ .

By Levy's theorem and Lemma 6.2,

$$\frac{d\tilde{\mathsf{P}}_{t \wedge \tau_n}}{d\mathsf{P}_{t \wedge \tau_n}} \xrightarrow[t \uparrow \infty]{} \mathsf{E}_{\mathsf{P}} \left( \frac{d\tilde{\mathsf{P}}_{\tau_n}}{d\mathsf{P}_{\tau_n}} \middle| \bigvee_{t \in [0, \infty)} \mathcal{F}_{t \wedge \tau_n} \right) = \frac{d\tilde{\mathsf{P}}_{\tau_n}}{d\mathsf{P}_{\tau_n}} \quad \mathsf{P}\text{-a.s.}$$

Thus, (43) for  $t \in [0, \infty)$  implies (43) also for  $t = \infty$ . This concludes the proof.  $\square$

For any  $t \in [0, \infty]$  let  $\tilde{\mathsf{Q}}_t$  denote the absolutely continuous part of the measure  $\tilde{\mathsf{P}}_t$  with respect to  $\mathsf{P}_t$ . Let us note that  $\tilde{\mathsf{Q}}_t(\Omega) = 1 \iff \tilde{\mathsf{P}}_t \ll \mathsf{P}_t$ .

**Lemma 6.4.** *We have*

$$\frac{d\tilde{\mathsf{Q}}_t}{d\mathsf{P}_t} = Z_t \quad \mathsf{P}\text{-a.s.}, \quad t \in [0, \infty].$$

*Proof.* Let us fix  $t \in [0, \infty]$ . By Lemma 6.1 and Jessen's theorem (see [37, Th. 5.2.26]),

$$\frac{d\tilde{P}_{t \wedge \tau_n}}{dP_{t \wedge \tau_n}} \xrightarrow{n \uparrow \infty} \frac{d\tilde{Q}_t}{dP_t} \quad P\text{-a.s.}$$

Since the process  $Z$  is stopped after  $\zeta$ , we have  $Z_{t \wedge \tau_n} \rightarrow Z_t$   $P$ -a.s. as  $n \uparrow \infty$ . Now the statement follows from Lemma 6.3.  $\square$

*Proof of Theorem 2.1.* We prove the remark after Theorem 2.1 (from this remark Theorem 2.1 follows). By Lemma 6.4 we have

$$(Z_t)_{t \in [0, T]} \text{ is a } P\text{-martingale} \iff E_P \frac{d\tilde{Q}_T}{dP_T} = 1 \iff \tilde{P}_T \ll P_T \iff S > T \quad \tilde{P}\text{-a.s.},$$

where the last equivalence follows from the remark after Lemma 5.4. Assume first that  $P \neq \tilde{P}$  which is, by the occupation times formula, equivalent to  $\nu_L(b \neq 0) > 0$ . Let us recall that all points in  $J$  are non-separating in our case and that an endpoint of  $J$  is non-separating if and only if it is good in the sense of (24) and (26). Applying now remark (ii) following Theorem 5.5 we get that  $S > T$   $\tilde{P}$ -a.s. if and only if the coordinate process  $X$  does not explode under  $\tilde{P}$  at a bad endpoint of  $J$ . This gives the criterion in Theorem 2.1.

Finally, let us assume that  $\nu_L(b \neq 0) = 0$  (i.e.  $P = \tilde{P}$ ). Then, clearly, if  $l$  (resp.  $r$ ) is bad, then  $\tilde{s}(l) = -\infty$  (resp.  $\tilde{s}(r) = \infty$ ), hence  $X$  does not explode at  $l$  (resp. at  $r$ ) under  $\tilde{P}$ . This means that the criterion in Theorem 2.1 works also in the case  $P = \tilde{P}$ . This proves the theorem.  $\square$

*Proof of Theorem 2.3.* By Lemma 6.4 we have<sup>8</sup>

$$(Z_t)_{t \in [0, \infty)} \text{ is a u.i. } P\text{-martingale} \iff E_P \frac{d\tilde{Q}_\infty}{dP_\infty} = 1 \iff \tilde{P} \ll P \iff S = \delta \quad \tilde{P}\text{-a.s.},$$

where the last equivalence follows from Lemma 5.4. We can now apply Theorem 5.5 and remark (ii) after it, in much the same way as in the proof of Theorem 2.1, to express the condition  $S = \delta$   $\tilde{P}$ -a.s. in terms of the functions  $\mu$ ,  $\sigma$ , and  $b$ . In the case  $P \neq \tilde{P}$  (i.e.  $\nu_L(b \neq 0) > 0$ ), we find that  $S = \delta$   $\tilde{P}$ -a.s. if and only if either both endpoints of  $J$  are good or  $\tilde{P}$ -almost all paths of  $X$  converge to a good endpoint of  $J$  as  $t \uparrow \zeta$ . In the latter possibility the other endpoint of  $J$  may be (and, actually, will be) bad. The former possibility is condition (D) in Theorem 2.3. It follows from Proposition A.3 that the latter possibility is described by conditions (B) and (C). Finally, condition (A) constitutes the remaining case  $\nu_L(b \neq 0) = 0$ . This concludes the proof.  $\square$

## APPENDIX

Here we state some well-known results about the behaviour of solutions of one-dimensional SDEs with the coefficients satisfying the Engelbert–Schmidt conditions (see e.g. [4, Sec. 4.1]) that are extensively used in Sections 5.2 and 6.

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)})$  be the canonical filtered space described in Section 5.2,  $X$  the coordinate process on  $\Omega$ , and  $\zeta$  the explosion time of  $X$ . Let the measure  $P$  on  $(\Omega, \mathcal{F})$  be the distribution of a (unique in law) solution of SDE (4), where the coefficients  $\mu, \sigma$  are Borel functions  $J \rightarrow \mathbb{R}$  satisfying conditions (5)–(6).

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<sup>8</sup>“U.i.” stands for “uniformly integrable”.

Consider the sets

$$\begin{aligned} A &= \left\{ \zeta = \infty, \limsup_{t \rightarrow \infty} X_t = r, \liminf_{t \rightarrow \infty} X_t = l \right\}, \\ B_r &= \left\{ \zeta = \infty, \lim_{t \rightarrow \infty} X_t = r \right\}, \\ C_r &= \left\{ \zeta < \infty, \lim_{t \uparrow \zeta} X_t = r \right\}, \\ B_l &= \left\{ \zeta = \infty, \lim_{t \rightarrow \infty} X_t = l \right\}, \\ C_l &= \left\{ \zeta < \infty, \lim_{t \uparrow \zeta} X_t = l \right\}. \end{aligned}$$

For  $a \in J$  define the stopping time

$$\tau_a = \inf\{t \in [0, \infty) : X_t = a\} \quad (\inf \emptyset := \infty).$$

**Proposition A.1.** *For any  $a \in J$  we have  $\mathbb{P}(\tau_a < \infty) > 0$ .*

**Proposition A.2.** *Either  $\mathbb{P}(A) = 1$  or  $\mathbb{P}(B_r \cup B_l \cup C_r \cup C_l) = 1$ .*

Let  $s$  denote the scale function of diffusion (4) and  $\rho$  the derivative of  $s$  (see (16) and (18)).

**Proposition A.3.** *(i) If  $s(r) = \infty$ , then  $\mathbb{P}(B_r \cup C_r) = 0$ .*

*(ii) Assume that  $s(r) < \infty$ . Then either  $\mathbb{P}(B_r) > 0$ ,  $\mathbb{P}(C_r) = 0$  or  $\mathbb{P}(B_r) = 0$ ,  $\mathbb{P}(C_r) > 0$ .<sup>9</sup>*

Moreover, we have

$$\mathbb{P}\left(\lim_{t \uparrow \zeta} X_t = r, X_t > a \ \forall t \in [0, \zeta)\right) > 0$$

for any  $a < x_0$ .

Proposition A.3 concerns the behaviour of the solution of SDE (4) at the endpoint  $r$ . Clearly, it has its analogue for the behaviour at  $l$ .

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<sup>9</sup>See (20) or (21) for a necessary and sufficient condition for  $\mathbb{P}(B_r) = 0$ ,  $\mathbb{P}(C_r) > 0$ . We therefore get that a necessary and sufficient condition for  $\mathbb{P}(B_r) > 0$ ,  $\mathbb{P}(C_r) = 0$  consists of  $s(r) < \infty$  and  $\frac{s(r)-s}{\rho\sigma^2} \notin L^1_{\text{loc}}(r-)$ .

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